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Griffith singularities in the random-field Ising model

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Abstract. The D-dimensional ferromagnetic Ising model with weak Gaussian random fields is considered. In dimensions $D < 3$ due to rare large-scale thermal excitations (large-spin clusters with magnetization opposite to the ferromagnetic background) in the low-temperature region $h_0^2 \ll T \ll 1$ (where h_0 is the characteristic value of the field) the free energy is shown to contain a non-analytic contribution of the form $\exp[-(\mathrm{const}/2h_0^2)(h_0^2/T)^{(3-D)/2}]$.

There are few reliable statements for the problem of the random-field Ising model. According to simple physical arguments by Imry and Ma (1975) one would expect that the lower critical dimension is two. Indeed, if we try to reverse a large region of the linear size L there are two competing effects: the gain in energy due to the alignment with the random magnetic field, which scales as $L^{D/2}$, and the loss of energy due to the creation of an interface, which scales as $L^{(D-1)}$. At dimensions two or less the two effects are comparable and no spontaneous magnetization should be present. On the other hand, at dimensions greater than two, this effect should not destroy the long-range order and a ferromagnetic transition should be present. This naive (but physically correct) argument was confirmed by a rigorous proof by Imbrie (1984).

On the other hand, a perturbative study of the transition shows that, as far as the leading infrared divergences are concerned, the strange phenomenon of a dimensional reduction is present, and the critical exponents of the system in dimensions D are the same as those of the ferromagnetic system without random fields in dimension $d = D - 2$ (Young 1977). This result would imply that the lower critical dimension is three, in contradiction with the rigorous results.

Actually, the procedure of summing the leading infrared divergences could give the correct result only if the Hamiltonian has only one minimum in the presence of the magnetic field. In this case, the dimensional reduction can rigorously be shown to be exact, by using supersymmetric arguments (Parisi and Sourlas 1979, Parisi 1987).

However, as soon as the temperature is smaller than the transition temperature of the system without a magnetic field, there are values of the magnetic field for which the free energy has more than one minimum. In this situation there is no reason to believe that the supersymmetric approach should give the correct results, and therefore the dimensional reduction is not founded. This is not surprising, because the dimensional reduction completely misses the appearance of Griffith's singularities (Griffith 1969).

Recently, it has been shown that the existence of more that one solution of the stationary equations in the presence of the external field is related in the replica approach to the existence of new solutions of the mean-field equations in replica space which are not invariant under translations and rotations in replica space (translation invariance and replica symmetry is recovered by considering the set of all possible solutions of this kind (Parisi and Dotsenko 1992).

In this paper using simple physical arguments the origin of the Griffith singularities in the thermodynamical functions in the low-temperature (ordered) phase in the temperature region $h_0^2 \ll T \ll 1$ for the dimensions $D < 3$ will be demonstrated. This non-perturbative contribution to the thermodynamics will be shown to come from rare, large-spin clusters having characteristic size $\sim \sqrt{T}/h_0$ with magnetization opposite to the ferromagnetic background, which are the *local* minima of the free energy.

The model under consideration is defined by the Hamiltonian,

$$H = -\sum_{\langle i \neq j \rangle}^{N} \sigma_i \sigma_j - \sum_i h_i \sigma_i \tag{1}$$

where the Ising spins $\{\sigma_i = \pm 1\}$ are placed in the vertices of some D-dimensional lattice with the ferromagnetic interaction between the nearest neighbours, and the quenched random fields $\{h_i\}$ are described by the symmetric Gaussian distribution

$$P[h_i] = \prod_i^N \left[\frac{1}{\sqrt{2\pi h_0^2}} \exp\left\{ -\frac{h_i^2}{2h_0^2} \right\} \right] \qquad h_0 \ll 1 \,. \tag{2}$$

If the dimensions of the system are larger than two, then the ground-state spin configuration is ferromagnetic. The thermal excitations are the spin clusters with the magnetization opposite to the background. If the linear size L of such a cluster is big, then (in the continuous limit) the energy of this thermal excitation could be estimated as follows:

$$E(L) \simeq L^{D-1} - V(L) \tag{3}$$

where

$$V(L) = \int_{|x|<L} \mathrm{d}^D x \, h(x) \,. \tag{4}$$

The statistical distribution of the energy function $V(L)$ (which is the energy of the spin cluster of the size L in the random field $h(x)$) is

$$P[V(L)] = \int \mathrm{D}h(x) \exp\left\{ -\frac{1}{2h_0^2} \int \mathrm{d}^D x \, h^2(x) \right\} \prod_L \left[\delta\left(\int_{|x|<L} \mathrm{d}^D x \, h(x) - V(L) \right) \right] \tag{5}$$

(here, and in what follows, all kinds of the pre-exponential factors are omitted). For future calculations it will be more convenient to deal with the quenched function $V(L)$ instead of $h(x)$. One can easily derive an explicit expression for the distribution function $P[V(L)]$, (5) (for simplicity the parameter L is first taken to be discrete):

$$\begin{aligned} P[V(L)] = {} & \left[\prod_x \int_{-\infty}^{+\infty} \mathrm{d}h(x) \right] \left[\prod_i \int_{-\infty}^{+\infty} \mathrm{d}\xi_i \right] \\ & \times \exp\left\{ -\frac{1}{2h_0^2} \int \mathrm{d}^D x \, h^2(x) + \mathrm{i} \sum_i \xi_i \left(\int_{|x|<L_i} \mathrm{d}^D x \, h(x) - V(L_i) \right) \right\} \\ & \times \left(\prod_i \int_{-\infty}^{+\infty} \mathrm{d}\xi_i \right) \exp\left(-\mathrm{i} \sum_i \xi_i V(L_i) \right) \left[\prod_x \int_{-\infty}^{+\infty} \mathrm{d}h(x) \right] \end{aligned}$$

$$\times \exp\left\{-\frac{1}{2h_0^2}\int \mathrm{d}^D x\, h^2(x) + \mathrm{i}\sum_{i=1}^{\infty}\int_{L_i<|x|<L_{i+1}} \mathrm{d}^D x\, h(x)\left[\sum_{j=i}^{\infty}\xi_j\right]\right\}$$

$$= \left(\prod_i \int_{-\infty}^{+\infty} \mathrm{d}\xi_i\right) \exp\left\{-\mathrm{i}\sum_i \xi_i V(L_i)\right) - \frac{1}{2}h_0^2 \sum_{l=1}^{\infty}(L_{i+1}^D - L_i^D)\left[\sum_{j=i}^{\infty}\xi_j\right]^2\right\}$$

$$= \exp\left\{-\frac{1}{2h_0^2}\sum_i \frac{[V(L_{i+1}) - V(L_i]^2}{L_{i+1}^D - L_i^D}\right\}. \tag{6}$$

Making L continuous again, one finally gets

$$P[V(L)] \simeq \exp\left\{-\frac{1}{2h_0^2}\int \mathrm{d}L \frac{1}{L^{D-1}}\left(\frac{\mathrm{d}V(L)}{\mathrm{d}L}\right)^2\right\}. \tag{7}$$

Since the probability of the flips of big spin clusters is exponentially small, their contributions to the partition function could be assumed to be independent (it is assumed that such clusters are non-interacting, being very far from each other). Then, their contribution to the total free energy could be obtained from the statistical averaging of the free energy of one isolated cluster:

$$\Delta F = -T\left[\prod_L \int \mathrm{d}V(L)\right] P[V(L)] \log\left[1 + \int_1^{\infty} \mathrm{d}L \exp\left\{\beta(V(L) - L^{D-1})\right\}\right]. \tag{8}$$

Here the factor under the logarithm is the partition function obtained as a sum over all the sizes of the flipped cluster (the factor '1' is the contribution of the ordered state which is the state without the flipped cluster).

The idea of the calculations of the above free energy is in the following. Since at dimensions $D > 2$ the energy $E(L) = L^{D-1} - V(L)$ is, on average, the growing function of L, it would be reasonable to expect that the deep local minima (if any) of this function are well separated and the values of the energies at these minima, on average, grow with the size L. For that reason, let us assume that the leading contribution in the integration over the sizes of the clusters in (8) comes only from *one* (if any) deepest local minimum of the function $L^{D-1} - V(L)$ (for a given realization of the quenched function $V(L)$).

Again, in view of the fact that the energy $E(L) = L^{D-1} - V(L)$ is, on average, the growing function of L, the sufficient condition for existence of a minimum somewhere above a given size L is

$$\frac{\mathrm{d}V(L)}{\mathrm{d}L} > (D-1)L^{D-2}. \tag{9}$$

Using the above assumptions, the contribution to the free energy from the flipped clusters (8) could be estimated as follows:

$$\Delta F \simeq -T\int_1^{\infty} \mathrm{d}L \int_{-\infty}^{+\infty} \mathrm{d}V\, P_L(V) P\left(\frac{\mathrm{d}V(L)}{\mathrm{d}L} > (D-1)L^{D-2}\right)$$
$$\times \log\left[1 + \exp\left\{\beta(V - L^{D-1})\right\}\right] \tag{10}$$

where $P_L(V)$ is the probability of a given value of the energy V at a given size L, and $P(\mathrm{d}V(L)/\mathrm{d}L > (D-1)L^{D-2})$ is the probability that the condition (9) is satisfied at the unit length at the given size L.

According to (4): $\langle\langle V^2(L)\rangle\rangle \simeq h_0^2 L^D$ (for large values of L). Since the distribution $P_L(V)$ should be expected to be Gaussian, one gets

$$P_L(V) \simeq \exp\left\{-\frac{V^2}{2h_0^2 L^D}\right\}. \tag{11}$$

Note that the above result could also be obtained by integrating the general distribution function $P[V(L)]$, equation (7), over all the 'trajectories' $V(L)$ with the fixed value $V(L) = V$ at the given length L.

The value of the probability $P(\mathrm{d}V(L)/\mathrm{d}L > (D-1)L^{D-2})$ could also be obtained by integrating $P[V(L)]$ over all the functions $V(L)$ conditioned by $\mathrm{d}V(L)/\mathrm{d}L > (D-1)L^{D-2}$ (at the given value of L). It is clear, however, that with the exponential accuracy the result of such integration is defined just by the lower bound $(D-1)L^{D-2}$ for the derivative $\mathrm{d}V(L)/\mathrm{d}L$ (at the given length L) in (7). Therefore, one gets

$$P\left(\frac{\mathrm{d}V(L)}{\mathrm{d}L} > (D-1)L^{D-2}\right) \simeq \exp\left\{-\frac{1}{2h_0^2L^{D-1}}((D-1)L^{D-2})^2\right\}$$
$$= \exp\left\{-\frac{(D-1)^2L^{D-3}}{2h_0^2}\right\}. \tag{12}$$

Note the important property of the energy $E(L)$, which follows from (11)–(12): although at dimensions $D > 2$ the function $E(L)$ grows with L, the probability to find a local minimum of this function at dimensions $D < 3$ also grows with L. It is the competition of these two effects which produces the non-trivial contribution to be calculated below.

In the limit of low temperatures, $\beta \gg 1$ (although still $T \gg h_0^2$), the contribution to the free energy, (10), could be divided into two separate parts:

$$\Delta F = \Delta F_1 + \Delta F_2 \simeq -T\int_1^\infty \mathrm{d}L \int_{V>L^{D-1}} \mathrm{d}V \exp\left\{-\frac{V^2}{2h_0^2L^D} - \frac{(D-1)^2L^{D-3}}{2h_0^2}\right\}$$
$$\times \log\left(1+\exp\left\{\beta(V-L^{D-1})\right\}\right)$$
$$-T\int_1^\infty \mathrm{d}L \int_{V<L^{D-1}} \mathrm{d}V \exp\left\{-\frac{V^2}{2h_0^2L^D} - \frac{(D-1)^2L^{D-3}}{2h_0^2}\right\}$$
$$\times \log\left(1+\exp\left\{\beta(V-L^{D-1})\right\}\right). \tag{13}$$

The first one is the contribution from the minima which have negative energies (the excitations which produce the *gain* in energy with respect to the ordered state). Here the leading contribution in the integration over V comes from the limit $V = L^{D-1}$, and in the leading order one gets

$$\Delta F_1 \sim -T\int_1^\infty \mathrm{d}L \exp\left\{-\frac{L^{D-2}}{2h_0^2} - \frac{(D-1)^2L^{D-3}}{2h_0^2}\right\}. \tag{14}$$

At dimensions $D > 2$ the leading contribution to ΔF_1 comes from $L \sim 1$ and this leads us back to the Imry and Ma (1975) arguments that there are no flipped big-spin clusters which would produce the gain in energy with respect to the ordered state.

The second contribution in (13) comes from the local minima which have positive energies. These could contribute to the free energy only as thermal excitations at non-zero temperatures. In the limit of low temperatures ($\beta \gg 1$) one could approximate

$$\log(1+\exp\{\beta(V-L^{D-1})\}) \simeq \exp\{-\beta(L^{D-1}-V)\} \tag{15}$$

where $L^{D-1} > V$. Then, for ΔF_2 one gets

$$\Delta F_2 \simeq -T\int_1^\infty \mathrm{d}L \int_{-\infty}^{L^{D-1}} \mathrm{d}V \exp\left\{-\frac{V^2}{2h_0^2L^D} - \frac{(D-1)^2L^{D-3}}{2h_0^2} + \beta V - \beta L^{D-1}\right\}. \tag{16}$$

The main contribution in this integral also comes from the 'trivial' region $L \sim 1$ and $V \sim \beta h_0^2$, which corresponds to the 'elementary excitations' at scales of the lattice spacing.

However, if the temperature is not too low, $\beta h_0^2 \ll 1$ and $D < 3$, there exists another non-trivial contribution which comes from the vicinity of the saddle point:

$$V_* = (\beta h_0^2) L_*^D$$
$$L_* = \sqrt{\frac{(D-1)(3-D)}{2\beta h_0^2}} \gg 1 \tag{17}$$

which is separated from the region $L \sim 1, V \sim \beta h_0^2$ by the big barrier. Note that the condition of integration in (16), $V_* \ll L_*^{D-1}$, according to (17) is satisfied for $L_* \ll 1/\beta h_0^2$, which is correct only if $\beta h_0^2 \ll 1$.

For the contribution to the free energy at this saddle point one gets

$$\Delta F_2 \sim \exp\left\{-\frac{\text{const}}{2h_0^2}(\beta h_0^2)^{(3-D)/2}\right\} \tag{18}$$

where

$$\text{const} = \frac{1}{2}(D+1)(D-1)^{(D-1)/2}\left(\frac{2}{3-D}\right)^{(3-D)/2} \tag{19}$$

The result (18) demonstrates that in addition to the usual thermal excitations in the vicinity of the ordered state (which could be taken into account by the traditional perturbation theory), due to the interaction with the random fields there exist essentially non-perturbative large-scale thermal excitations which produce exponentially small non-analytic contribution to the thermodynamics. These excitations are big-spin clusters with the magnetization opposite to the background which are the *local* energy minima. At finite temperature such that $h_0^2 \ll T \ll 1$ the characteristic size of the clusters giving the leading contribution to the free energy is $L_* \sim \sqrt{T}/h_0 \gg 1$.

This phenomenon, although it seems to produce a negligibly small contribution to the thermodynamical functions, could be extremely important for understanding the dynamical relaxation processes. The big clusters with reversed magnetization being the local minima are separated from the ground state by big energy barriers, and this could produce essential slowing down of the relaxation (see e.g. Nowak and Usadel 1991). In particular, the characteristic 'saddle-point' clusters (17), with size $L_*(T) \sim \sqrt{T}/h_0 \gg 1$, are separated from the ground state by the energy barrier of the order of $V_* \sim (\beta h_0^2)^{-(D-2)/2} \gg 1$, and the corresponding characteristic relaxation time at low temperatures should be expected to be exponentially large:

$$\tau(T) \sim \exp\left\{\beta(\beta h_0^2)^{-(D-2)/2}\right\} \gg 1\,. \tag{20}$$

However, to describe the time asymptotics of the relaxation processes one needs to know the *spectrum* of the relaxation times (or the energy barriers), and this would require more special consideration.

Unfortunately, the results obtained in this paper cannot be directly applied for dimensions $D = 3$, which appears to be marginal for the considered phenomena (at dimensions $D > 3$ this sort of non-perturbative effects are absent). At $D = 3$ none of those simple estimates for the energies and probabilities of the cluster excitations which have been used in this paper (in particular, (12)) work, and much more detailed analysis is required.

On the other hand, it seems quite reasonable to expect that the results obtained are correct at dimensions $D = 2$ regardless of the fact that the long-range order is not stable there. The point is that at $D = 2$ the correlation length at which the long-range order is destroyed is exponentially large in the parameter $1/h_0$, while the characteristic size of the spin clusters considered here is only the power of the parameter $1/h_0$. Therefore, at the scales at which the Griffith singularities (18) appear, the system is still effectively ordered at $D = 2$.

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